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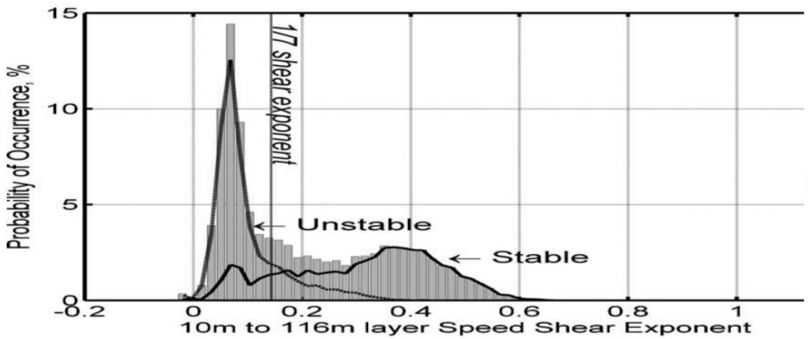

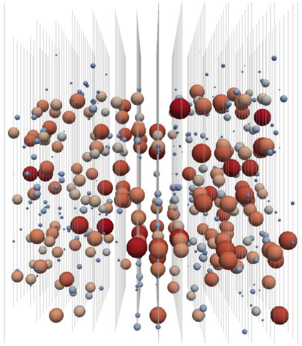

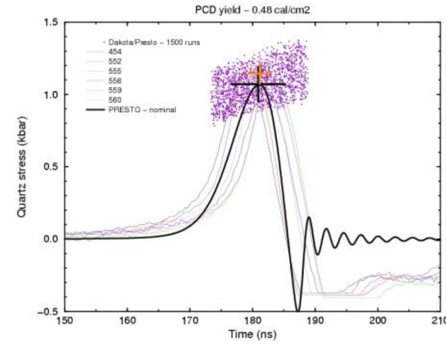

Dakota Sensitivity Analysis and Uncertainty Quantification, with Examples

U.S. DEPARTMENT OF ENERGY

NNSA National Nuclear Security Administration

SAND2014-3134P

Dakota Uncertainty Quantification (UQ)

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- UQ goals and examples
- Select Dakota examples for UQ:
  - Monte Carlo sampling
  - Local and global reliability
  - Polynomial chaos expansions / stochastic collocation
  - Mixed aleatory-epistemic approaches
  - Probabilistic design
- Dakota primarily focuses on forward propagation
  - Secondarily on estimating parameter uncertainty given data
  - Not on processing experimental data to calculate uncertainties

Drivers for Dakota UQ

Current Dakota research and development largely focuses on efficient UQ for large-scale engineering analyses.

DOE in general, ASC V&V in particular, are:

- Responding to shift from test-based to modeling and simulation-based design and certification
- Demanding risk-informed decision-making using credible M&S:
  - *Predictive simulations:* verified, validated for application domain of interest
  - *Quantified margins and uncertainties:* random variability effect is understood, *best estimate with uncertainty prediction* for decision-making

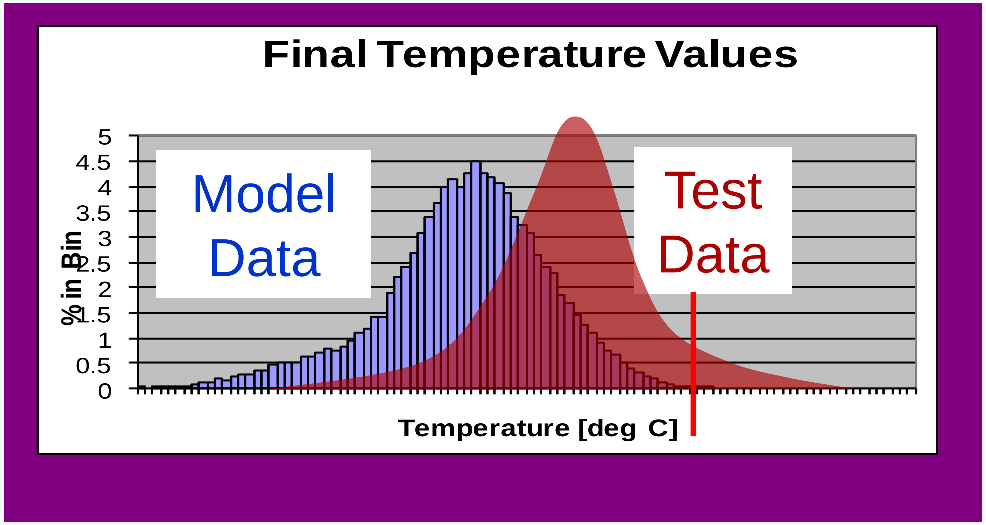
Why Perform Uncertainty Quantification?

- What? Determine variability, distributions, statistics of code outputs, given uncertainty in input factors
- Why? Assess likelihood of typical or extreme outcomes. Given input uncertainty…
  - Determine mean or median performance of a system
  - Assess variability in model response
  - Find probability of reaching failure/success criteria (reliability metrics)
  - Assess range/intervals of possible outcomes
- Assess how close uncertainty-endowed code predictions are to
  - Experimental data
    (validation, is model sufficient *for the intended application?)*
  - Performance expectations or limits
    (quantification of margins and uncertainties; QMU)

Many Potential Uncertainties in Simulation and Validation

Parameterized…:

- physics/science parameters
- statistical variation, inherent randomness
- model form / accuracy
- material properties
- manufacturing quality
- operating environment, interference
- initial, boundary conditions; forcing
- geometry / structure / connectivity
- experimental error (measurement error, measurement bias)
- numerical accuracy (mesh, solvers); approximation error
- human reliability, subjective judgment, linguistic imprecision



The effect of these on model outputs should be integral to an analyst's deliverable: *best estimate PLUS uncertainty!*

Forward Parametric Uncertainty Quantification

- **Identify and characterize uncertain variables (may not be normal, uniform)**
- ***Forward propagate: quantify the effect that (potentially correlated) uncertain (nondeterministic) input variables have on model output:***

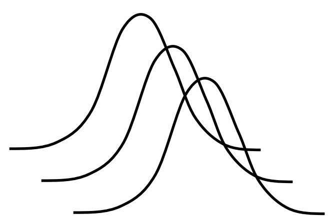
<u>Input Variables *u*</u>
(physics parameters, geometry, initial and boundary conditions)

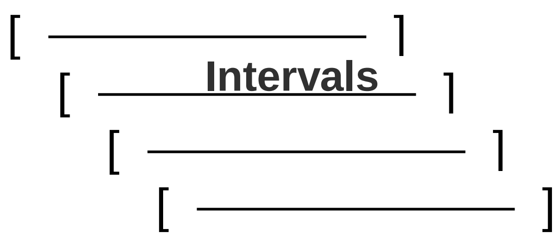
→ **Computational Model** →

Variable Performance Measures *f(u)*

Uncertainties on inputs

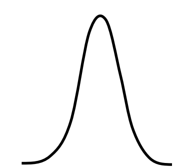
- Parameterized distributions: normal, uniform, gumbel, etc.
- Means, standard deviations
- PDF, CDF from data
- Intervals
- Belief structures





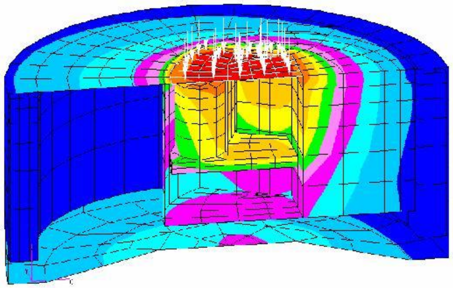
Uncertainties on outputs

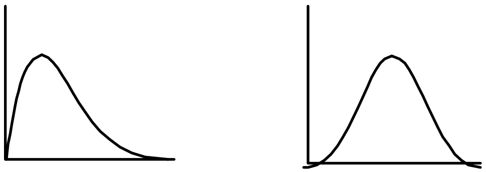
- Means, standard deviations
- Probabilities
- Reliabilities
- PDF, CDF
- Intervals
- Belief, plausibility

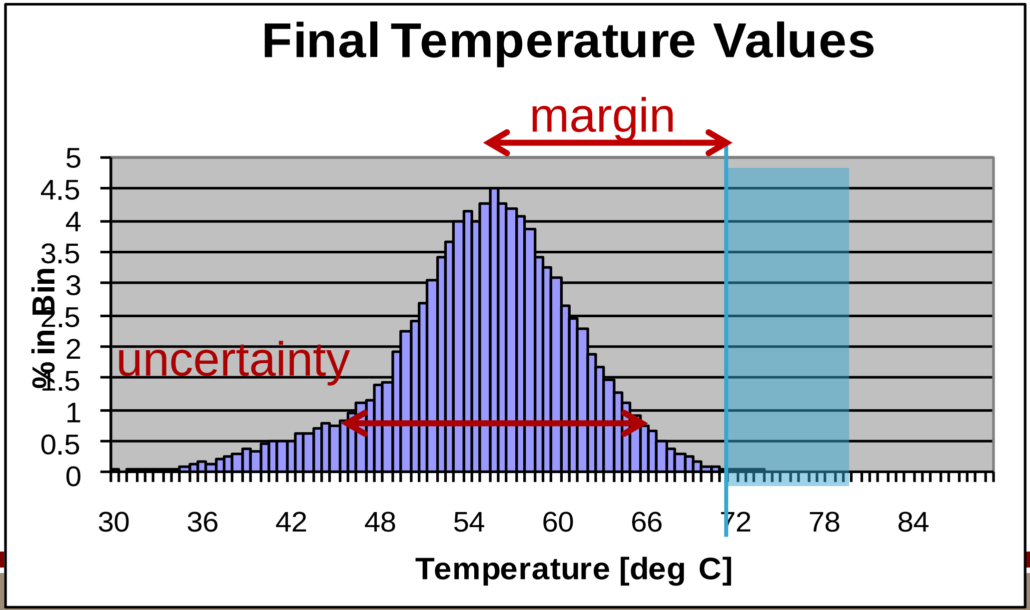


Example:
Thermal Uncertainty Quantification

- Device subject to heating (experiment or computational simulation)
- Uncertainty in composition/ environment (thermal conductivity, density, boundary), parameterized by $u_1, \ldots, u_N$
- Response temperature $f(u)=T(u_1, \ldots, u_N)$ calculated by heat transfer code







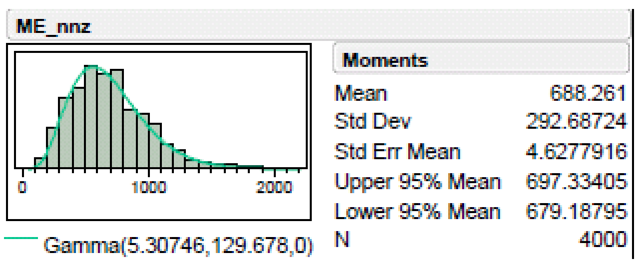
Given distributions of $u_1,\ldots,u_N$, UQ methods calculate statistical info on outputs:

- Mean(T), StdDev(T), Probability($T \geq T_{\text{critical}}$)
- Probability distribution of temperatures
- *Correlations (trends) and sensitivity of temperature*

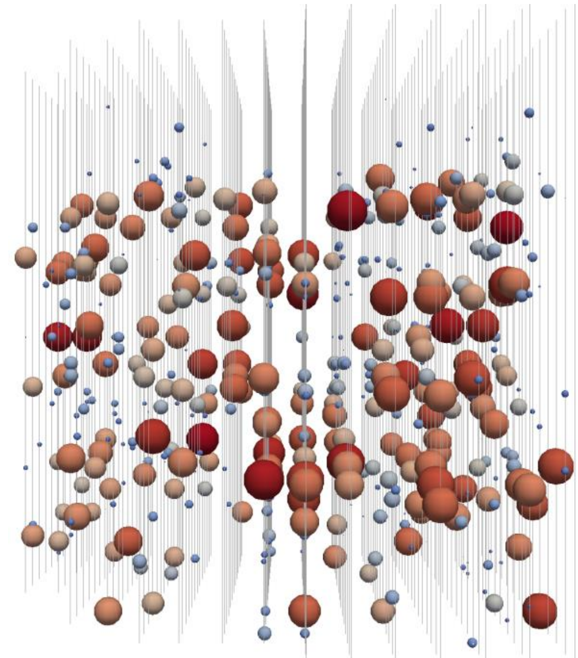
Example: Uncertainty in Boiling Rate in Nuclear Reactor Core

Method	ME_nnz		ME_meannz		ME_max	
	Mean	Std Dev	Mean	Std Dev	Mean	Std Dev
LHS (40)	651.225	297.039	127.836	27.723	361.204	55.862
LHS (400)	647.33	286.146	127.796	25.779	361.581	51.874
LHS (4000)	688.261	292.687	129.175	25.450	364.317	50.884
PCE (Θ(2))	687.875	288.140	129.151	25.7015	364.366	50.315
PCE (Θ (3))	688.083	292.974	129.231	25.3989	364.310	50.869
PCE (Θ (4))	688.099	292.808	129.213	25.4491	364.313	50.872

mean and standard deviation of key metrics

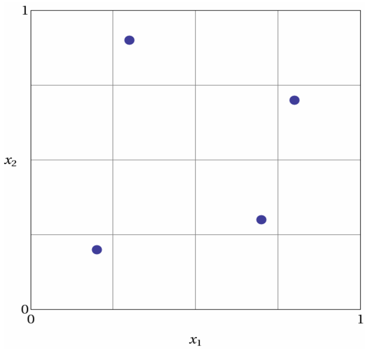


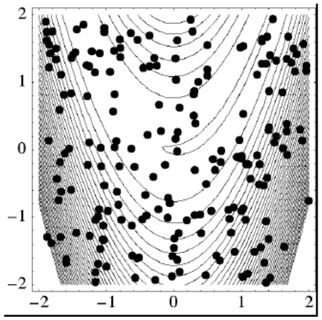
normally distributed inputs need not give rise to normal outputs…

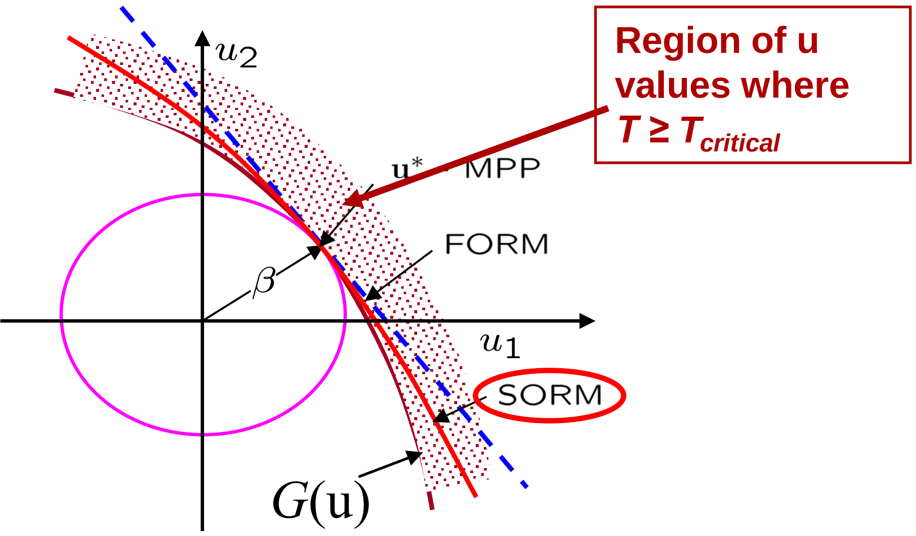


anisotropic uncertainty distribution in boiling rate throughout quarter core model

Three Core Dakota UQ Methods



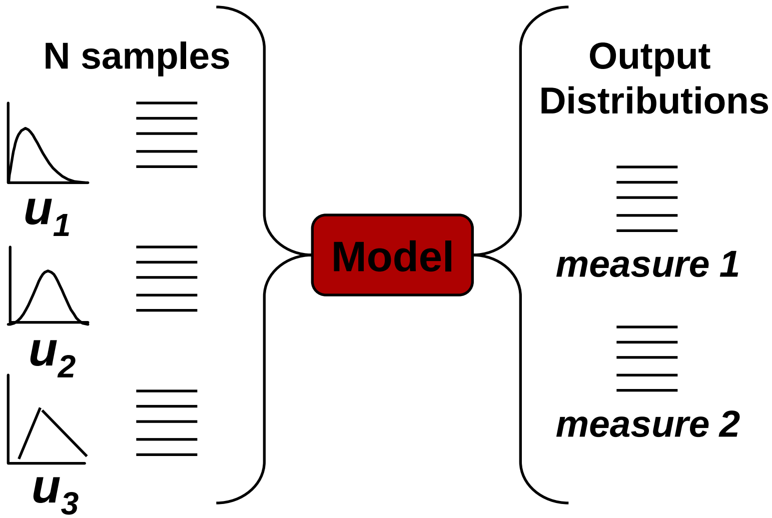




- Sampling (Monte Carlo, Latin hypercube): robust, easy to understand, slow to converge / resolve statistics
- Reliability: good at calculating probability of a particular behavior or failure / tail statistics; efficient, some methods are only local
- Stochastic Expansions (PCE/SC global approximations): efficient tailored surrogates, statistics often derived analytically, *far more efficient than sampling for reasonably smooth functions*

Black-box UQ Workhorse: Random Sampling Methods

Given distributions of $u_1,\dots,u_N$, sampling-based methods calculate sample statistics, e.g., on temperature $T(u_1,\dots,u_N)$:



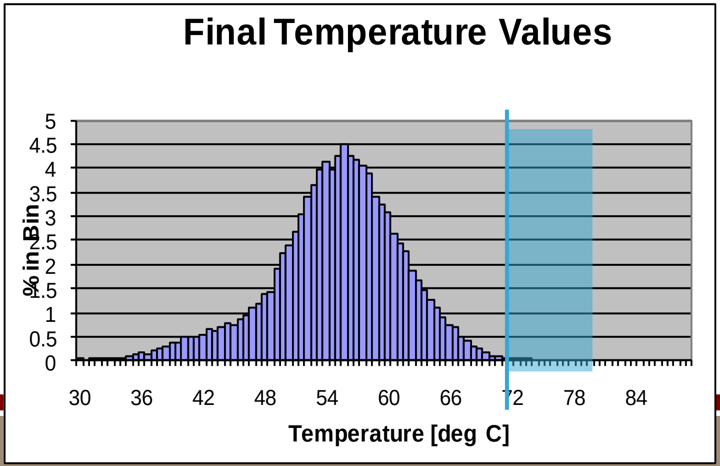
- **sample mean**

$$\bar{T} = \frac{1}{N}\sum_{i=1}^{N} T(u^i)$$

- **sample variance**

$$T_{\sigma^2} = \frac{1}{N}\sum_{i=1}^{N}\left[T(u^i) - \bar{T}\right]^2$$

- **full PDF(probabilities)**



- Monte Carlo sampling, Quasi-Monte Carlo
- Centroidal Voroni Tessalation (CVT)
- Latin hypercube (stratified) sampling: better convergence; stability across replicates

Robust, but slow convergence: $O(N^{-1/2})$, independent of dimension *(in theory)*

Example:
Cantilever Beam UQ with Sampling

- Dakota study with LHS
- Determine mean system response, variability, margin to failure given
  - Density P ~ Normal(500, 30)
  - Young's modulus E ~ Normal(2.9e7, 2.e6)
  - Horizontal load X ~ Normal(50, 3)
  - Vertical load Y ~ Normal(100, 6)
  - *(Dakota supports a wide range of distribution types)*
- Hold width and thickness at 1.0, L at 5.
- Compute with respect to thresholds with `probability_levels` or `response_levels`
  - What is the probability(stress < 10000)?
  - What is the probability(mass < 1.5)?
  - What is the probability(displacement < 0.002)?

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  - *(Dakota supports a wide range of distribution types)*
- Hold width and thickness at 1.0, L at 5.
- Compute with respect to thresholds with `probability_levels` or `response_levels`
  - What is the probability(stress < 10000)? ~0.9 for uniform, 0.99 for normal
  - What is the probability(mass < 1.5)? ~0.6 for uniform, 0.8 for normal
  - What is the probability(displacement < 0.002)? ~0.6 for uniform, 0.7 for normal

Dakota Input: LHS Sampling for Cantilever Beam

```
method
  sampling
  sample_type lhs
  samples = 100
  seed = 3845
  num_probability_levels = 17 17 17
  probability_levels =
   .001 .01 .05 .1 .15 .2 .3 .4 .5 .6 .7 .8 .85 .9 .95 .99 .999
   .001 .01 .05 .1 .15 .2 .3 .4 .5 .6 .7 .8 .85 .9 .95 .99 .999
   .001 .01 .05 .1 .15 .2 .3 .4 .5 .6 .7 .8 .85 .9 .95 .99 .999
  cumulative distribution


variables
  active uncertain
  continuous_design = 3
    upper_bounds = 1.2 1.2 6.0
    lower_bounds = 0.8 0.8 4.0
    descriptors    "w"     "t"     "L"
  uniform_uncertain = 4
    upper_bounds = 600. 35.E+6 60. 120.
    lower_bounds = 400. 23.E+6 40. 80.
    descriptors    'p'    'E'    'X'   'Y'

…
responses
  response_functions = 3
  descriptors = 'mass' 'stress' 'displacement'
  no_gradients no_hessians
```

Dakota Output:
LHS Sampling for Cantilever Beam

- Moments and confidence intervals

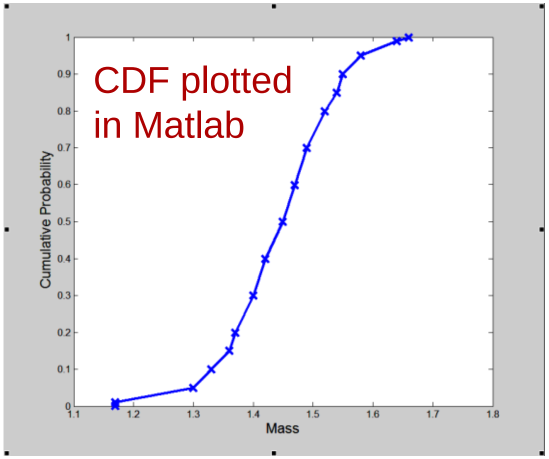
```
Statistics based on 100 samples:

Moment-based statistics for each response function:
                            Mean           Std Dev          Skewness          Kurtosis
          mass  1.4460475709e+00  8.8239262134e-02 -1.6051074470e-01  2.5955294928e-01
        stress  8.9986343326e+04  4.0344159128e+03  6.4230716871e-02  1.0335094626e-01
  displacement  1.9378806350e-03  1.6660999428e-04  5.5574418567e-01  5.8860476955e-01

95% confidence intervals for each response function:
                    LowerCI_Mean      UpperCI_Mean    LowerCI_StdDev    UpperCI_StdDev
          mass  1.4285389869e+00  1.4635561549e+00  7.7474676187e-02  1.0250536737e-01
        stress  8.9185827682e+04  9.0786858970e+04  3.5422447886e+03  4.6866811355e+03
  displacement  1.9048215975e-03  1.9709396725e-03  1.4628471549e-04  1.9354670764e-04
```

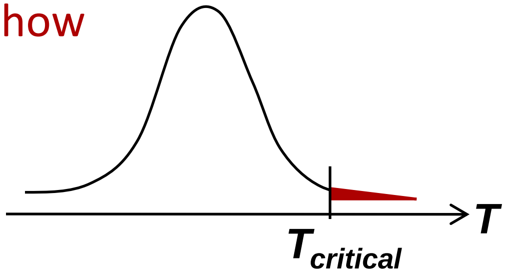
- CDF (and PDF) data

```
Level mappings for each response function:
Cumulative Distribution Function (CDF) for mass:
     Response Level  Probability Level  Reliability Index
     --------------  -----------------  -----------------
   1.1683297300e+00   1.0000000000e-03
   1.1683297300e+00   1.0000000000e-02
   1.2951111800e+00   5.0000000000e-02
   1.3316578300e+00   1.0000000000e-01
   1.3559746900e+00   1.5000000000e-01
   1.3734105800e+00   2.0000000000e-01
   1.4003385200e+00   3.0000000000e-01
   1.4245467700e+00   4.0000000000e-01
```



Challenge: Calculating Potentially Small Probability of Failure

- Given uncertainty in materials, geometry, and environment, how to determine likelihood of failure: *Probability*($T \geq T_{critical}$)?
- Perform 10,000 LHS samples and count how many exceed threshold;
  (better) perform adaptive importance sampling



Mean value: make a linearity (and possibly normality) assumption and project; great for many parameters with efficient derivatives!

$$\mu_T = T(\mu_u)$$

$$\sigma_T = \sum_i \sum_j Cov_u(i,j) \frac{dg}{du_i}(\mu_u) \frac{dg}{du_j}(\mu_u)$$

Reliability: directly determine input variables which give rise to failure behaviors by solving an optimization problem for a most probable point (MPP) of failure

$$\text{minimize} \quad u^T u$$

$$\text{subject to} \quad T(u) = T_{critical}$$

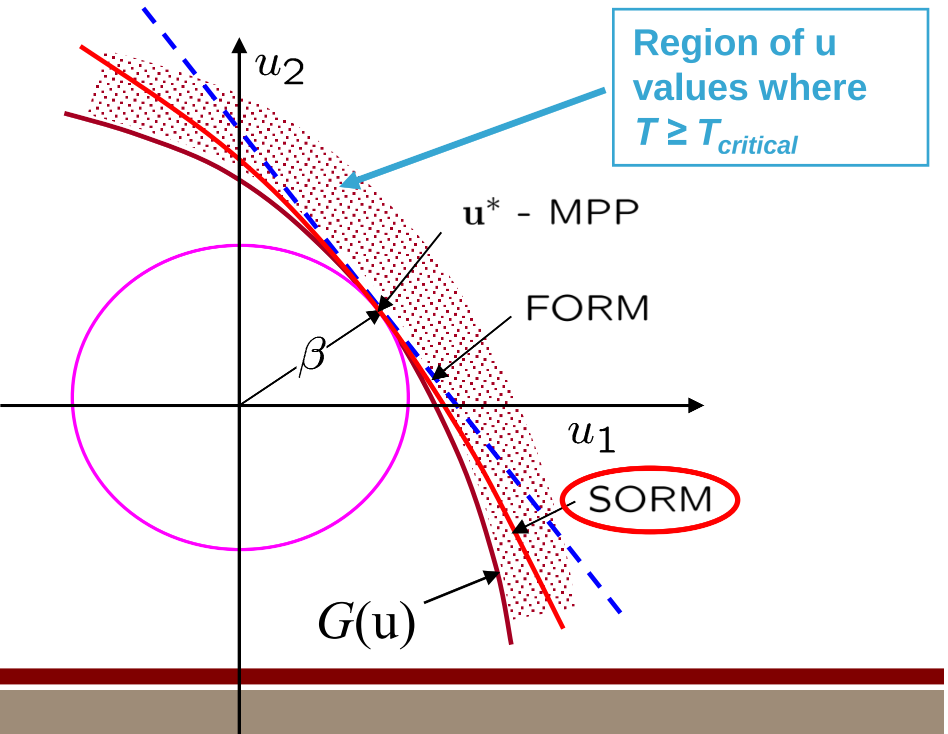
Analytic Reliability: MPP Search

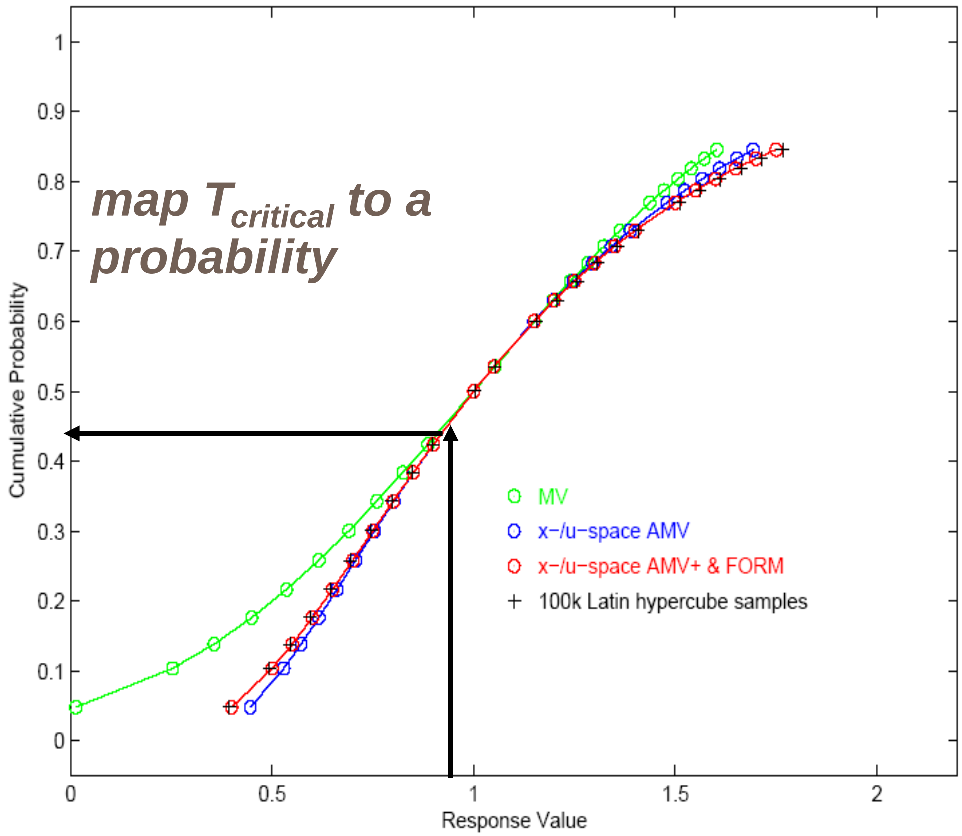
Perform optimization in uncertain variable space to determine Most Probable Point (of response or failure occurring) for *G(u) = T(u)*.

Reliability Index Approach (RIA)

$$\begin{array}{ll}\text{minimize} & \mathbf{u}^T\mathbf{u} \\ \text{subject to} & G(\mathbf{u}) = \bar{z}\end{array}$$

All the usual nonlinear optimization tricks apply…

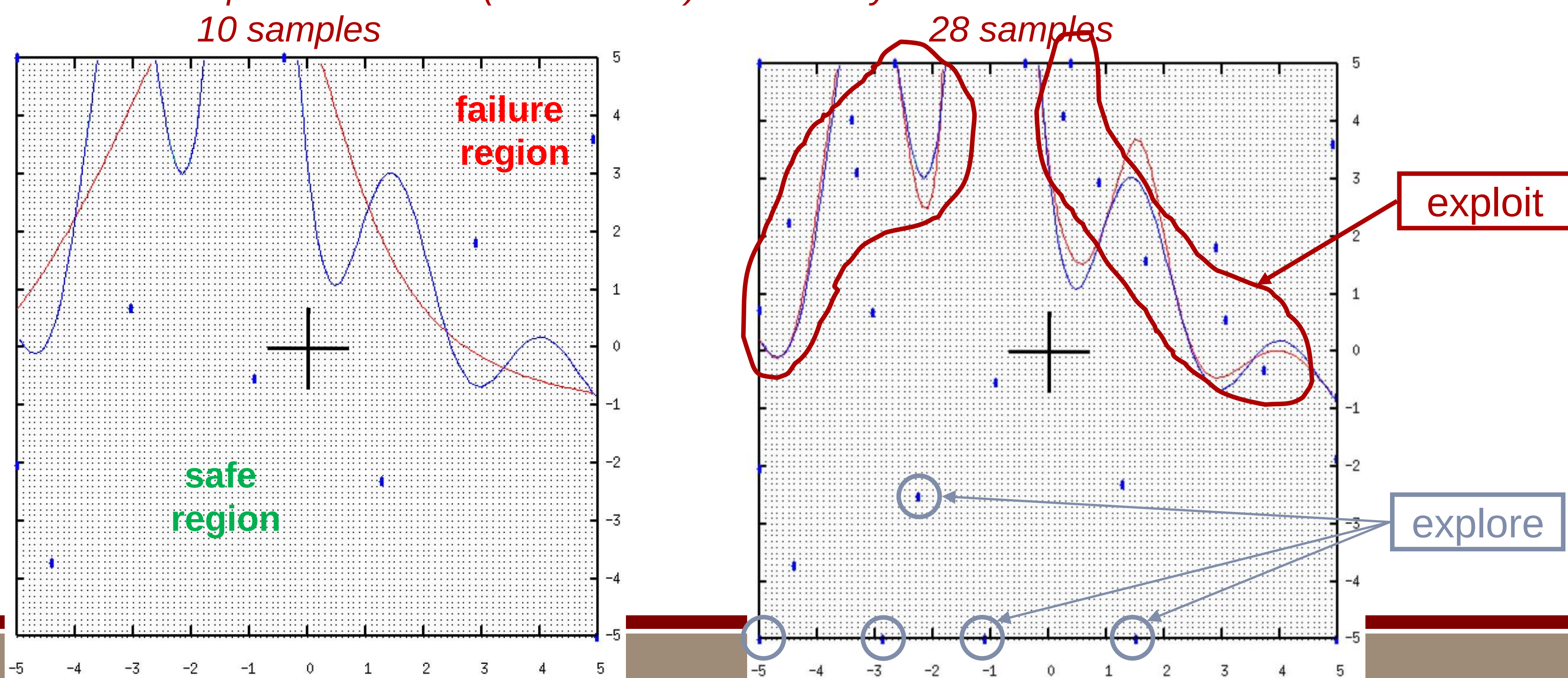




Efficient Global Reliability Analysis Using Gaussian Process Surrogate + MMAIS

- Efficient global optimization (EGO)-like approach to solve optimization problem
- Expected feasibility function: balance exploration with local search near failure boundary to refine the GP
- Cost competitive with best local MPP search methods, yet better probability of failure estimates; addresses nonlinear and multimodal challenges

Gaussian process model (level curves) of reliability limit state with



Generalized Polynomial Chaos Expansions (PCE)

Approximate response with Galerkin projection using multivariate orthogonal polynomial basis functions defined over standard random variables

$$R = \sum_{j=0}^{P} \alpha_j \Psi_j(\xi)$$

$$\boxed{R(\xi) \approx f(u)}$$

$$\boxed{\alpha_j = \frac{\langle R, \Psi_j \rangle}{\langle \Psi_j^2 \rangle} = \frac{1}{\langle \Psi_j^2 \rangle} \int_\Omega R \, \Psi_j \, \varrho(\xi) \, d\xi}$$

- Intrusive or non-intrusive
- Wiener-Askey Generalized PCE: optimal basis selection leads to exponential convergence of statistics

Distribution	Density function	Polynomial	Weight function	Support range
Normal	$\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$	Hermite $He_n(x)$	$e^{\frac{-x^2}{2}}$	$[-\infty, \infty]$
Uniform	$\frac{1}{2}$	Legendre $P_n(x)$	1	$[-1, 1]$
Beta	$\frac{(1-x)^\alpha (1+x)^\beta}{2^{\alpha+\beta+1} B(\alpha+1, \beta+1)}$	Jacobi $P_n^{(\alpha,\beta)}(x)$	$(1-x)^\alpha (1+x)^\beta$	$[-1, 1]$
Exponential	e^{-x}	Laguerre $L_n(x)$	e^{-x}	$[0, \infty]$
Gamma	$\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}$	Generalized Laguerre $L_n^{(\alpha)}(x)$	$x^\alpha e^{-x}$	$[0, \infty]$

- Can also numerically generate basis orthogonal to empirical data (PDF/histogram)

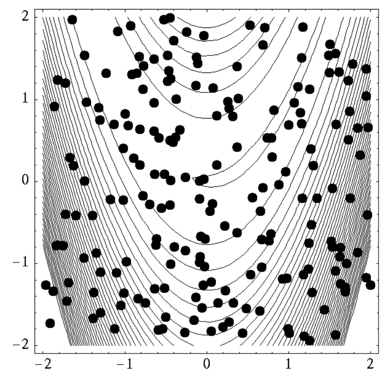
Sample Designs to Form Polynomial Chaos or Stochastic Collocation Expansions

Random sampling: PCE

Expectation (sampling):
- **Sample w/i distribution of x**
- **Compute expected value of product of R and each Y_j**

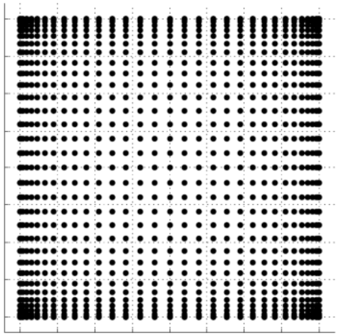
Linear regression ("point collocation"):

$$\boxed{\Psi\alpha = R}$$

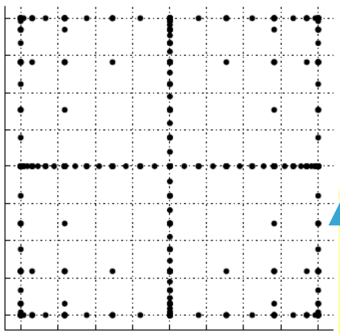


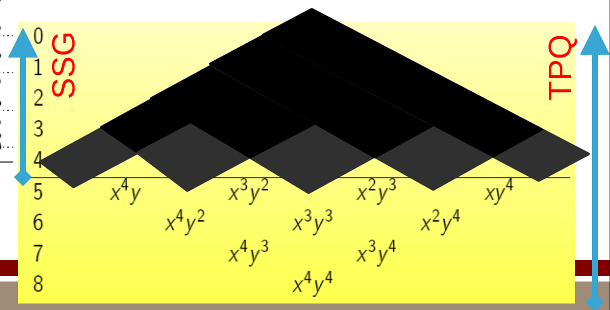
Tensor-product quadrature: PCE/SC

Tensor product of 1-D integration rules, e.g., Gaussian quadrature



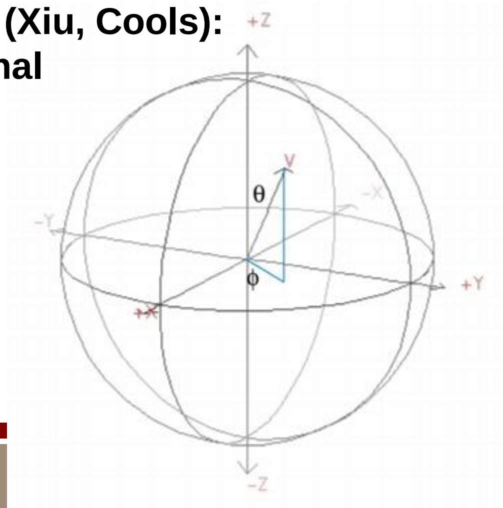
Smolyak Sparse Grid: PCE/SC





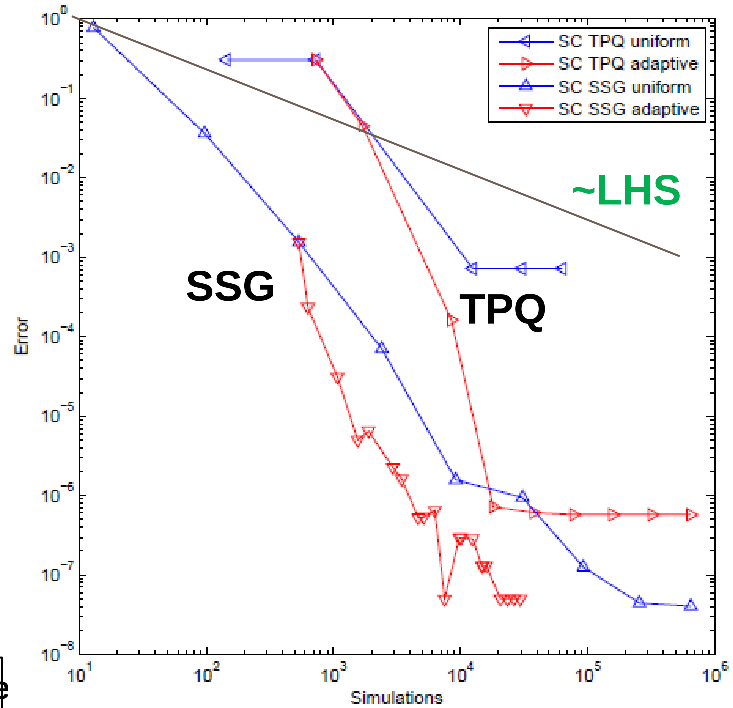
Cubature: PCE

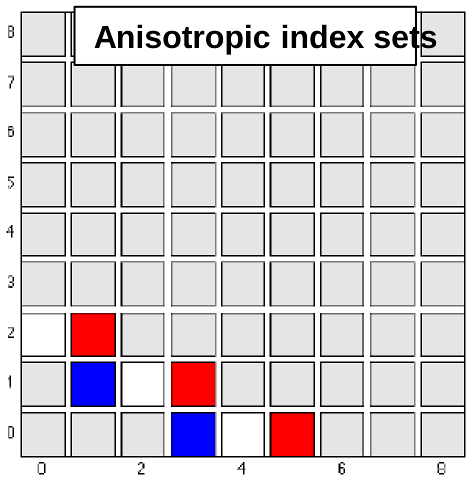
Stroud and extensions (Xiu, Cools): optimal multidimensional integration rules

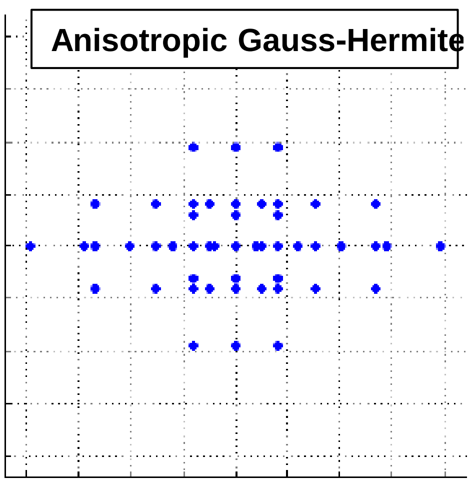


Adaptive PCE/SC: Emphasize Key Dimensions

- Judicious choice of new simulation runs
- Uniform p-refinement
  - Stabilize 2-norm of covariance
- Adaptive p-refinement
  - Estimate main effects/VBD to guide
- h-adaptive: identify important regions and address discontinuities
- h/p-adaptive: p for performance; h for robustness







Changes for Reliability, PCE

```
method,
      local_reliability
       num_probability_levels = 17 17 17
        probability_levels =
        .001 .01 .05 .1 .15 .2 .3 .4 .5 .6 .7 .8
.85 .9 .95 .99 .999
        .001 .01 .05 .1 .15 .2 .3 .4 .5 .6 .7 .8
.85 .9 .95 .99 .999
        .001 .01 .05 .1 .15 .2 .3 .4 .5 .6 .7 .8
.85 .9 .95 .99 .999
        cumulative distribution
responses
  response_functions = 3
  descriptors = 'mass' 'stress' 'displacement'
  numerical_gradients
    method_source dakota
    interval_type central
    fd_gradient_step_size = 0.0001
 no_hessians
```

```
method,
     polynomial_chaos
     sparse_grid_level = 2
     sample_type lhs
     samples = 10000
     seed = 8572
     num_probability_levels = 17 17 17
     probability_levels =
      .001 .01 .05 .1 .15 .2 .3 .4 .5 .6 .7 .8
.85 .9 .95 .99 .999
      .001 .01 .05 .1 .15 .2 .3 .4 .5 .6 .7 .8
.85 .9 .95 .99 .999
      .001 .01 .05 .1 .15 .2 .3 .4 .5 .6 .7 .8
.85 .9 .95 .99 .999
      cumulative distribution
```

Uncertainty Quantification Research in Dakota: New algorithms bridge robustness/efficiency gap

Sandia National Laboratories

	Production	New	Under dev.	Planned	Collabs.
Sampling	**Latin Hypercube, Monte Carlo**	**Importance, Incremental**		**Bootstrap, Jackknife**	**FSU**
Reliability	***Local:* Mean Value, First-order & second-order reliability methods (FORM, SORM)**	***Global:* Efficient global reliability analysis (EGRA)**	**gradient-enhanced**	**recursive emulation, TGP**	***Local:* Notre Dame, *Global:* Vanderbilt**
Stochastic expansion		**PCE and SC with uniform & dimension-adaptive p-/h-refinement**	**Local adapt refinement, gradient-enhanced, compr sens**	**Discrete rv, orthogonal least interp.**	**Stanford, Purdue**
Other probabilistic			**Rand fields/ stoch proc**	**Dimension reduction**	**Cornell, Maryland**
Epistemic	**Interval-valued/ Second-order prob. (nested sampling)**	**Opt-based interval estimation, Dempster-Shafer**	**Bayesian, discrete/ model form**	**Imprecise probability**	**LANL, UT Austin**
Metrics & Global SA	**Importance factors, Partial correlations**	**Main effects, Variance-based decomposition**		**Stepwise regression**	**LANL**

Research: Adaptive Refinement, Gradient Enhancement

Adv. Deployment

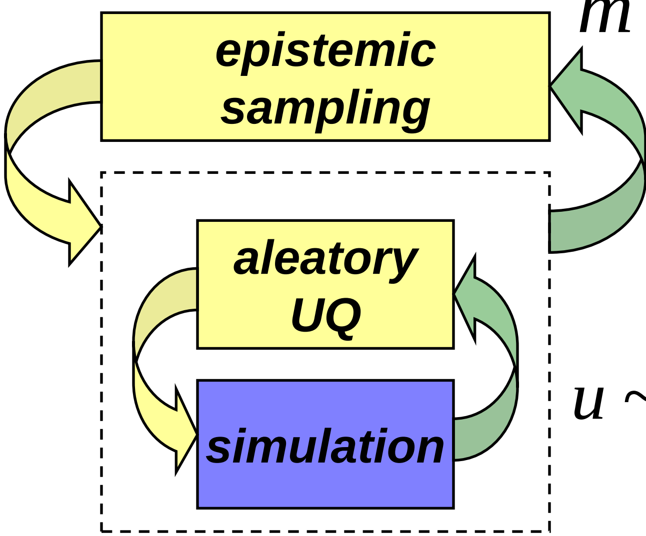
Fills Gaps

Aleatory/Epistemic UQ: Nested (“Second-order”)Approaches

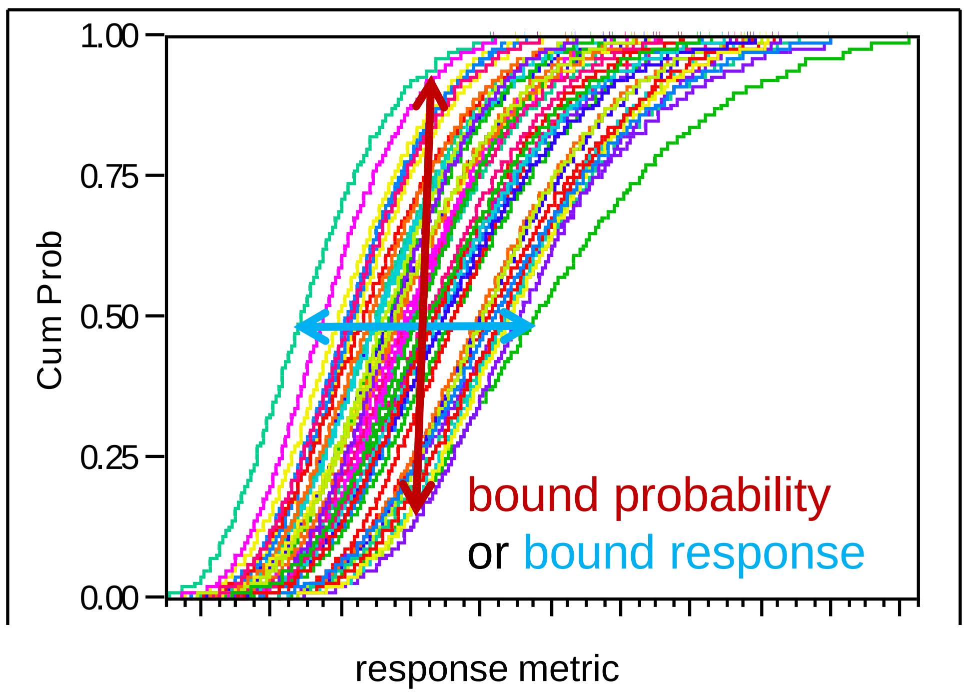
- Propagate over epistemic and aleatory uncertainty, e.g., UQ with bounds on the mean of a normal distribution (hyper-parameters)
- Typical in regulatory analyses (e.g., NRC, WIPP)
- Outer loop: epistemic (interval) variables, inner loop UQ over aleatory (probability) variables; *potentially costly, not conservative*
- If treating epistemic as uniform, do not analyze probabilistically!

50 outer loop samples:
50 aleatory CDF traces

$$m \in [L, U]$$



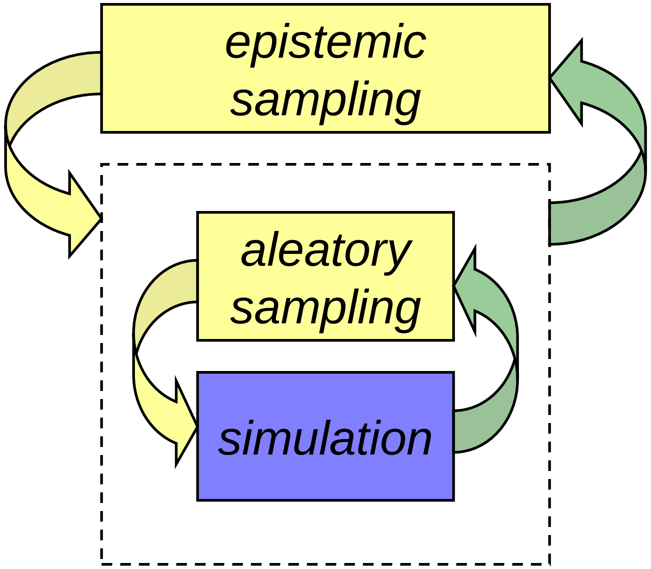
$$u \sim N(m, \sigma)$$



“Envelope” of CDF traces represents response epistemic uncertainty

Dakota Mixed UQ with Nested Model

- Two models, each with a different set of variables
- Outer method operates on nested model
- Inner method operates on simulation model



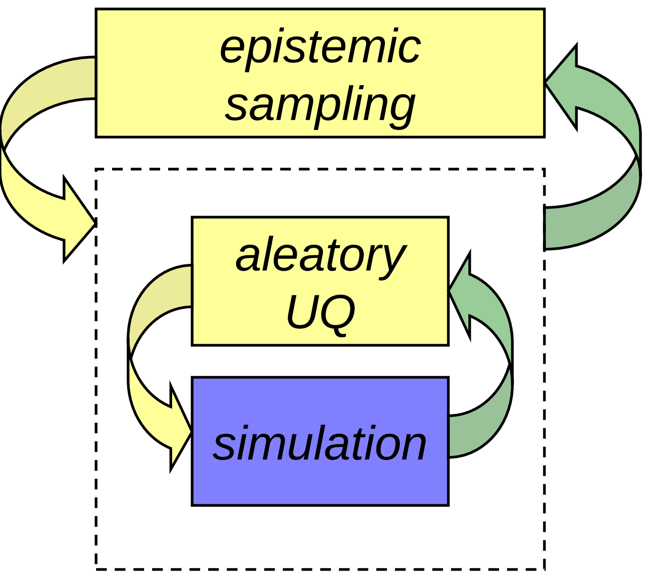

```
method
  id_method = 'EPISTEMIC'
  model_pointer = 'EPIST_M'
  sampling sample_type lhs
  samples = 5 seed = 12347

model,
  id_model = 'EPIST_M'
  nested
  variables_pointer   = 'EPIST_V'
  sub_method_pointer = 'ALEATORY'
  responses_pointer   = 'EPIST_R'
  primary_variable_mapping   = 'X'     'Y'
  secondary_variable_mapping = 'mean' 'mean'
  primary_response_mapping   = 1. 0. 0. 0. 0. 0. 0. 0. 0.
                               0. 0. 0. 0. 1. 0. 0. 0. 0.
                               0. 0. 0. 0. 0. 0. 0. 0. 1.

variables,
  id_variables = 'EPIST_V'
  interval_uncertain = 2
  num_intervals = 1 1
  interval_probabilities = 1.0 1.0
  upper_bounds = 600.  1200.
  lower_bounds = 400.  800.

responses,
  id_responses = 'EPIST_R'
  response_functions = 3
  descriptors ='mean_mass' '95th_perc_stress''95th_perc_disp'
  no_gradients no_hessians
```

Example Output: Intervals on Statistics



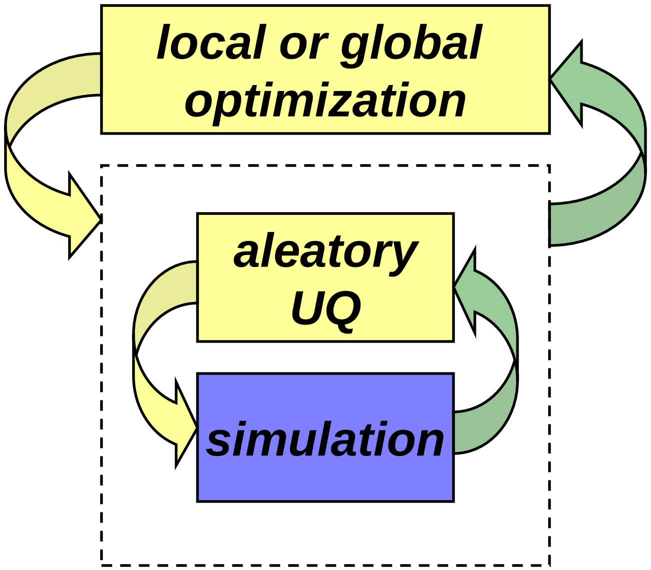

```
<<<<< Iterator nond_sampling completed.
<<<<< Function evaluation summary (ALEAT_I): 971 total (971
new, 0 duplicate)

Statistics based on 50 samples:

Min and Max values for each response function:
mean_wt:  Min = 9.5209117200e+00  Max = 9.5209117200e+00
ccdf_beta_s:  Min = 1.8001336086e+00  Max = 4.0744019409e+00
ccdf_beta_d:  Min = 1.9403177486e+00  Max = 3.7628144053e+00

Simple Correlation Matrix between input and output:
                  mean_wt  ccdf_beta_s  ccdf_beta_d
      X_mean  9.40220e-16 -6.38145e-01 -9.14016e-01
      Y_mean  1.38778e-15 -7.93481e-01 -4.39133e-01
       ….
```

Interval Estimation Approach (Probability Bounds Analysis)

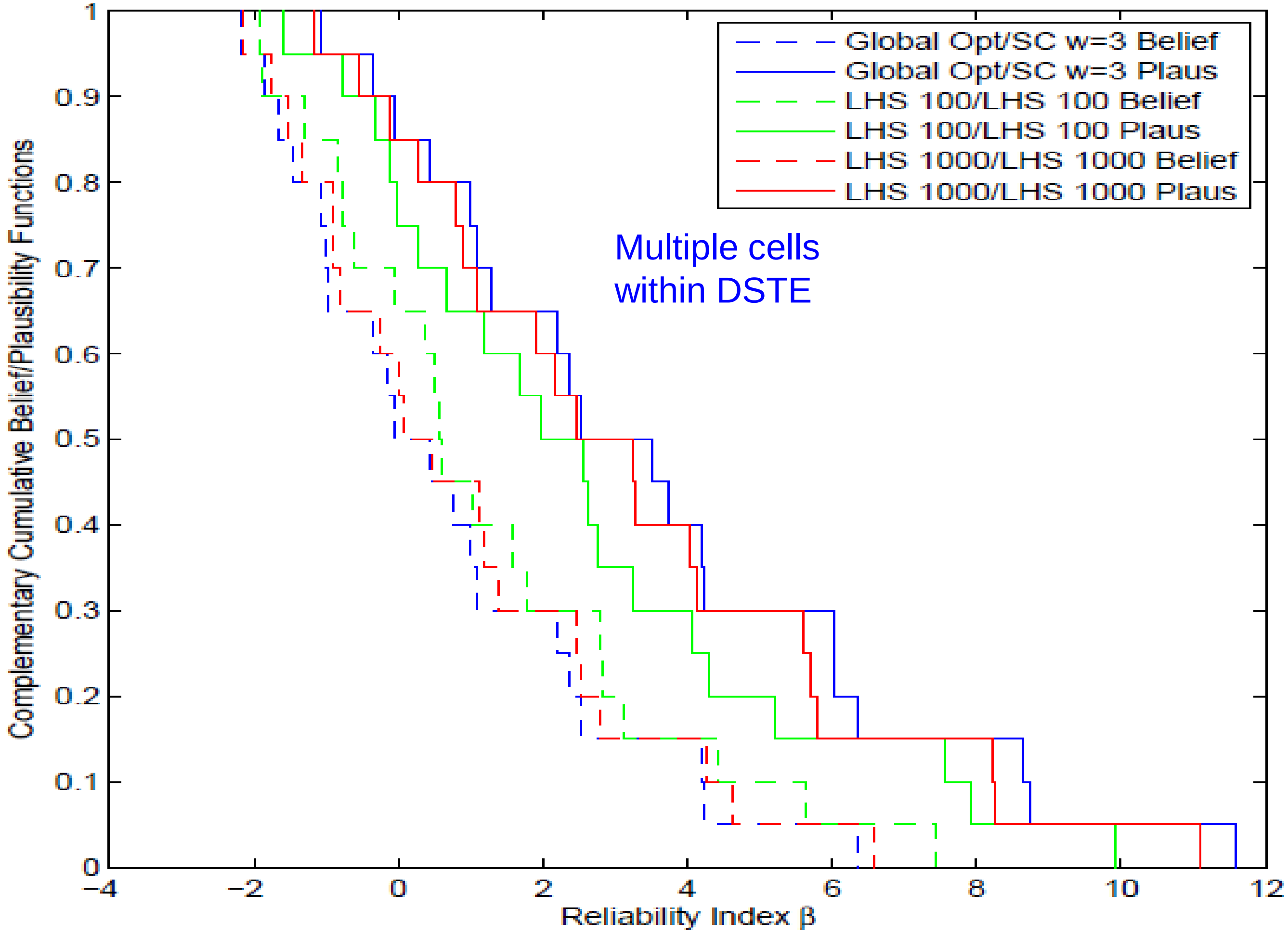


- *Propagate intervals through simulation code*
- Outer loop: determine interval on statistics, e.g., mean, variance
  - global optimization problem: find max/min of statistic of interest, given bound constrained interval variables
  - use EGO to solve 2 optimization problems with essentially one Gaussian process surrogate
- Inner loop: Use sampling, PCE, etc., to determine the CDFs or moments with respect to the aleatory variables

$$\min_{u_E} \; f_{STAT}(u_A \mid u_E)$$
$$u_{LB} \le u_E \le u_{UB}$$
$$u_A \sim F(u_A; u_E)$$

$$\max_{u_E} \; f_{STAT}(u_A \mid u_E)$$
$$u_{LB} \le u_E \le u_{UB}$$
$$u_A \sim F(u_A; u_E)$$

Interval Analysis can be Tractable for Large-Scale Apps



Converge to more conservative bounds with 10—100x less evaluations

Model Form UQ in Fluid/Structure Interactions

Low

Med

High

Discrete model choices for same physics:

- A clear hierarchy of fidelity (low to high)
- An ensemble of models that are all credible (lacking a clear preference structure)
  - With data: Bayesian model selection
  - Without data: epistemic model form uncertainty propagation

SA-RANS — KE-RANS-NBC — KE-RANS-DBC

- Combination:

Potential flow — Vortex lattice

SA-RANS — KE-RANS-NBC — KE-RANS-DBC

Smagorinsky-LES — Germano-LES

DNS

Horizontal Axis Wind Turbine   ***Vertical Axis Wind Turbine***

wind turbine applications

Multifidelity UQ using Stochastic Expansions

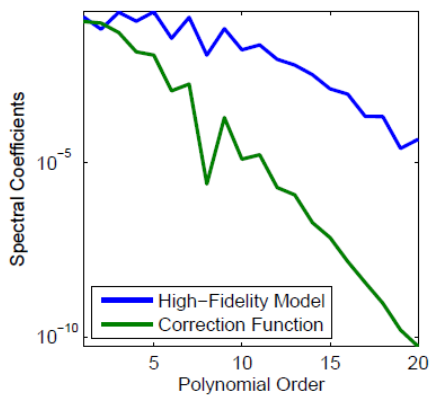
- High-fidelity simulations (e.g., RANS, LES) can be prohibitive for use in UQ
- Low fidelity "design" codes often exist that are predictive of basic trends
- Can we leverage LF codes w/i HF UQ in a rigorous manner? → global approxs. of model discrepancy

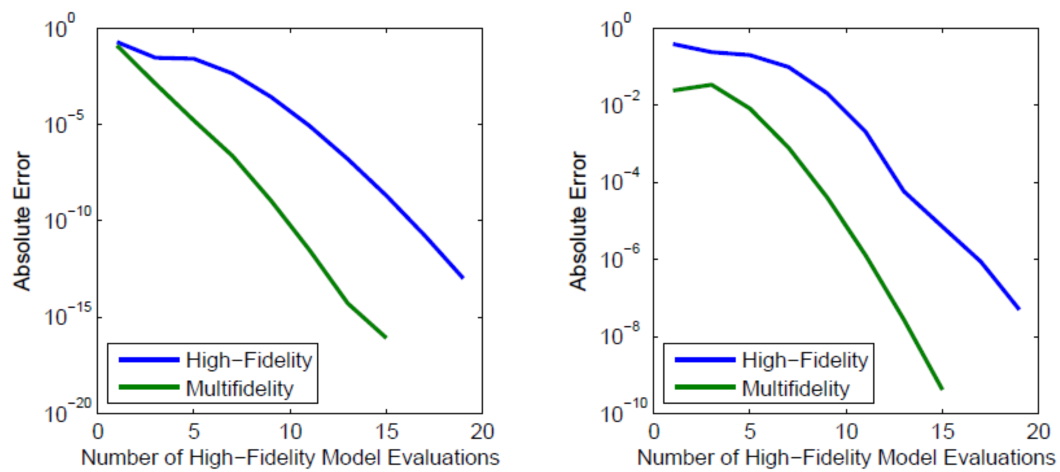
$$\hat{f_{hi}}(\xi) = \sum_{j=1}^{N_{lo}} f_{lo}(\xi_j) L_j(\xi) + \sum_{j=1}^{N_{hi}} \Delta f(\xi_j) L_j(\xi)$$

$N_{lo} >> N_{hi}$

$$R_{\text{high}}(\xi) = e^{-0.05\xi^2} \cos 0.5\xi - \underbrace{0.5e^{-0.02(\xi-5)^2}}_{\text{discrepancy}}$$

$$R_{\text{low}}(\xi) = e^{-0.05\xi^2} \cos 0.5\xi,$$





(a) Error in mean (b) Error in standard deviation

Low fidelity

CACTUS: Code for Axial and Crossflow TUrbine Simulation

High fidelity: DG formulation for LES
Full Computational Fluid Dynamics/
Fluid-Structure Interaction

Uncertainty Quantification not Addressed Here

- Efficient epistemic UQ [Dakota]
- Fuzzy sets (Zadeh)
- Imprecise Probability (Walley)
- Dempster-Shafer Theory of Evidence (Klir, Oberkampf, Ferson) [Dakota]
- Possibility theory (Joslyn)
- Probability bounds analysis (p-boxes)
- Info-gap analysis (Ben-Haim)

- Bayesian model calibration / inference via MCMC [Dakota]
- Other Bayesian approaches: Bayesian belief networks, Bayesian updating, Robust Bayes, etc.
- Scenario evaluation

(Some available in [Dakota])

Dakota UQ: Summary, Relevant Methods

- What? Understand code output uncertainty / variability
- Why? Risk-informed decisions with variability, possible outcomes
- How? What Dakota methods are relevant?

character	method class	problem character	variants
aleatory	probabilistic sampling	nonsmooth, multimodal, modest cost, # variables	Monte Carlo, LHS, importance
	local reliability	smooth, unimodal, more variables, failure modes	mean value and MPP, FORM/SORM,
	global reliability	nonsmooth, multimodal, low dimensional	EGRA
	stochastic expansions	nonsmooth, multimodal, low dimension	polynomial chaos, stochastic collocation
epistemic	interval estimation	simple intervals	global/local optim, sampling
	evidence theory	belief structures	global/local evidence
both	nested UQ	mixed aleatory / epistemic	nested

- See Dakota Usage Guidelines in User's Manual
- Analyze tabular output with third-party statistics packages

UQ References

- SAND report 2009-3055. “Conceptual and Computational Basis for the Quantification of Margins and Uncertainty” J. Helton.
- Helton, JC, JD Johnson, CJ Sallaberry, and CB Storlie. “Survey of Sampling-Based Methods for Uncertainty and Sensitivity Analysis”, *Reliability Engineering and System Safety* 91 (2006) pp. 1175-1209
- Helton JC, Davis FJ. Latin Hypercube Sampling and the Propagation of Uncertainty in Analyses of Complex Systems. *Reliability Engineering and System Safety* 2003;81(1):23-69.
- Haldar, A. and S. Mahadevan. *Probability, Reliability, and Statistical Methods in Engineering Design* (Chapters 7-8). Wiley, 2000.
- Eldred, M.S., "Recent Advances in Non-Intrusive Polynomial Chaos and Stochastic Collocation Methods for Uncertainty Analysis and Design," paper AIAA-2009-2274 in Proceedings of the 11th AIAA Non-Deterministic Approaches Conference, Palm Springs, CA, May 4-7, 2009.

- Dakota User’s Manual: Uncertainty Quantification Capabilities
- Dakota Theory Manual
- Corresponding Reference Manual sections